ON THE CONVERGENCE OF QUASILINEAR OBJECTS

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The games problem of the convergence of quasilinear objects with restrictions on the instantaneous values of the controlling forces is considered. It is shown that in regular situations the extremal strategies yield the saddle point of the game in question. An iterative method for constructing the extremal strategies is described.

1. Statement of the problem. Let the controlled objects be described by the differential equations

$$y^{\cdot} = A^{(1)}(t)\, y + B^{(1)}(t)\, u + \lambda f^{(1)}(y, t) \tag{1.1}$$

$$z^{\cdot} = A^{(2)}(t)\, z + B^{(2)}(t)\, v + \lambda f^{(2)}(z, t) \tag{1.2}$$

Here $y = \{y_1, \ldots, y_n\}$ is the n-dimensional vector of the phase coordinates of the pursuing object; $z = \{z_1, \ldots, z_n\}$ is the n-dimensional vector of the phase coordinates of the pursued object (target); $u = \{u_1, \ldots, u_r\}$, $v = \{v_1, \ldots, v_r\}$ are r-dimensional vector functions describing the forces controlling the pursuing and pursued objects, respectively; $A^{(j)}$ and $B^{(j)}$ are matrices of the appropriate dimensions which are continuous in t; $f^{(1)}(y, t)$ and $f^{(2)}(z, t)$ are vector functions continuous in t and twice continuously differentiable with respect to y and z for $y \in \Gamma_1$, $z \in \Gamma_2$, where Γ_1 and Γ_2 are some closed bounded domains; λ $(\lambda > 0)$ is a small parameter.

Let us consider the motion of the objects over a finite time interval $t_0 \leqslant t \leqslant \vartheta$. The controlling forces u and v are subject to the instantaneous restrictions $u[t] \in U^*$, $v[t] \in V^*$. The sets U^* and V^* of the vectors u and v are described by the inequalities

$$\| u[t] \| \leqslant \mu, \qquad \| v[t] \| \leqslant \nu \qquad (\mu, \nu = \text{const}) \tag{1.3}$$

The symbol $\| x \|$ here and everywhere below denotes the Euclidean norm of the vector x.

We define the cost of the game as the quantity

$$\gamma[\vartheta] = \| \{y[\vartheta]\}_m - \{z[\vartheta]\}_m \| \tag{1.4}$$

where $\{x\}_m$ is the vector consisting of the m first components of the vector x. The quantity $\gamma[\vartheta]$ estimates the distance between the objects at the final instant ϑ. The task of the pursuer is to minimize $\gamma[\vartheta]$; the task of the target (pursued object) is to maximize $\gamma[\vartheta]$.

Games convergence problems have been investigated by several authors (e.g. see [1-5]). Our purpose in the present study is to justify the extremal construction [4, 5] in the case of quasilinear systems with restrictions (1.3) imposed on the controlling forces. We shall make extensive use of the definitions and constructions of [4, 5]; for this reason we shall often omit relevant remarks concerning these constructions and the quantities occurring in them.

Let us assume that the pursuer and target know the realized values of $y\,[t]$ and $z\,[t]$ at each instant t and that the controlling forces are generated by the feedback principle, i.e. that the realized values of $u\,[t]$ and $v\,[t]$ at each instant t are generated on the basis of information on the quantities $y\,[t]$ and $z\,[t]$.

We define the strategy U (or V) of the players as the totality of sets $U^*\,(t,\,y,\,z,\,\lambda)$ (or $V^*\,(t,\,y,\,z,\,\lambda)$) consisting of r-dimensional vectors u (or v) which are associated with each possible position $\{t,\,y,\,z\}$. The realizations $u\,[t]$ (or $v[t]$) then satisfy the following requirements:

a) the inclusion

$$u\,[t] \in U^*\,([t,\,y\,[t],\,z\,[t],\,\lambda) \quad (\text{or } v\,[t] \in V^*\,(t,\,y\,[t],\,z\,[t],\,\lambda)) \qquad (1.5)$$

must be fulfilled for every t;

b) the function $u\,[t]$ (or $v\,[t]$) must be integrable over the time interval $t_0 \leqslant t \leqslant \vartheta$.

A strategy U defined by the sets $U^*\,(t,\,y,\,z,\,\lambda)$ (a strategy V defined by the sets $V^*\,(t,\,y,\,z,\,\lambda)$) will be called "permissible" if the totality of these sets satisfies the following conditions:

1) the inclusions

$$U^*\,(t,\,y,\,z,\,\lambda) \subset U^*\,(V^*\,(t,\,y,\,z,\,\lambda) \subset V^*)$$

are fulfilled;

2) the sets $U^*\,(t,\,y,\,z,\,\lambda)$ $(V^*\,(t,\,y,\,z,\,\lambda))$ are closed and convex;

3) the sets $U^*\,(t,\,y,\,z,\,\lambda)$ $(V^*\,(t,\,y,\,z,\,\lambda))$ for $\lambda \leqslant \lambda_0$ (λ_0 is sufficiently small) are semicontinuous above the inclusion as t, y and z vary in the neighborhood of each possible position.

Let us suppose that the first and second players have chosen some permissible strategies U and V. We define the solution of Eqs. (1.1) and (1.2) under controls $u \in U^*\,(t,\,y,\,z,\,\lambda)$, $v \in V^*\,(t,\,y,\,z,\,\lambda)$ (in the interval $t_1 \leqslant t \leqslant t_2$) as any absolutely continuous vector functions $y\,[t]$ and $z\,[t]$ which satisfy the equations

$$y^{\cdot}\,[t] = A^{(1)}\,(t)y\,[t] + B^{(1)}\,(t)u\,[t] + \lambda f^{(1)}\,(y\,[t],\;t)$$

$$z^{\cdot}\,[\,t] = A^{(2)}\,(t)z\,[t] + B^{(2)}\,(t)v\,[t] + \lambda f^{(2)}\,(z\,[t],\;t)$$

for almost all values $t \in [t_1,\;t_2]$. Here the vector functions $u\,[t]$ and $v\,[t]$ satisfy condition (1.5). Given solutions $y\,[t]$ and $z\,[t]$ can be called the "motions" of systems (1.1), (1.3) generated by the strategies U and V.

Let $(\gamma\,[\vartheta]\,|\,t_0,\,y_0,\,z_0,\,u,\,v)$ be the realization of the quantity $\gamma\,[\vartheta]$ (1.4) which coresonds to the initial position $t_0,\,y_0 \in \Gamma_1^\circ \subset \Gamma_1,\,z_0 \in \Gamma_2^\circ \subset \Gamma_2$ under the controls u and v.

Problem 1.1. We are to find that optimal strategy U° from among the permissible strategies U which satisfies the inequality

$$(\gamma\,[\vartheta]\,|\,t_0,\,y_0,\,z_0,\,U^\circ,\,v) \leqslant \min_U \sup_{v[t]} \inf_{y[t]} (\gamma\,[\vartheta]\,|\,t_0,\,y_0,\,z_0,\,U,\,v)$$

for any initial position $t_0,\;y_0,\,z_0\;\;(y_0 \in \Gamma_1^\circ,\,z_0 \in \Gamma_2^\circ,\,0 \leqslant t_0 < \vartheta)$.

Problem 1.2. We are to find that optimal strategy V° from among the permissible strategies V which satisfies the inequality

$$(\gamma\,[\vartheta]\,|\,t_0,\,y_0,\,z_0,\,u,\,V^\circ) \geqslant \max_V \inf_{u[t]} \sup_{z[t]} (\gamma\,[\vartheta]\,|\,t_0,\,y_0,\,z_0,\,u,\,V)$$

for any initial position t_0, y_0, z_0 $(y_0 \in \overset{\circ}{\Gamma}_1,\ z_0 \in \overset{\circ}{\Gamma}_2,\ 0 \leqslant t_0 < \vartheta)$.

2. Ancillary statement. Let us consider the controlled system

$$dx/d\tau = A(\tau)x + B(\tau)w + \lambda f(x, \tau) \tag{2.1}$$

Let the control $w(\tau)$ $(t \leqslant \tau \leqslant \vartheta)$ be subject to the restriction

$$\|w(\tau)\| \leqslant \zeta \qquad (\zeta = \text{const}) \tag{2.2}$$

Let us take an arbitrary m-dimensional unit vector l $(\|l\| = 1)$ and denote by $x(\tau; w)$ $(t \leqslant \tau \leqslant \vartheta)$ the motion of system (2.1) generated by some control $w(\tau)$ $(t \leqslant \tau \leqslant \vartheta)$ restricted by inequality (2.2) under the initial condition $\tau = t$, $x(t; w) = x$.

Now let us consider the problem of constructing a control $w^\circ(\tau)$ $(t \leqslant \tau \leqslant \vartheta)$ satisfying the condition

$$\rho[l, t, x, \lambda] = \max_{\|w\| \leqslant \zeta} l'\{x(\vartheta; w)\}_m = l'\{x^\circ(\vartheta; w^\circ)\}_m \tag{2.3}$$

Let

$$d\delta x/d\tau = A^\circ(\tau;\ l,\ t,\ x,\ \lambda)\,\delta x \tag{2.4}$$

be a system of equations in variations constructed for system (2.1) along the motion $x^\circ(\tau; w^\circ)$. Here the matrix A° is defined by the equation $A^\circ = A + A^{(0)}$, and the elements $a_{ij}^{(0)}$ of the matrix $A^{(0)}$ are defined by the relations

$$a_{ij}^{(0)}(\tau; l, t, x, \lambda) = \lambda \partial f_i(x^\circ(\tau; w^\circ), \tau) / \partial x_j$$

Let us denote the fundamental matrix of system (2.4) by $X[\vartheta, \tau;\ l, t, x, \lambda]$ $(X[\tau, \tau;\ l, t, x, \lambda] = E)$. Let the following conditions be fulfilled:

2.1. Every motion of the first-approximation system

$$dx/d\tau = A(\tau)x + B(\tau)w \tag{2.5}$$

generated by the control $w(\tau)$ $(\|w(\tau)\| \leqslant \zeta,\ t \leqslant \tau \leqslant \vartheta)$ under the initial condition $\tau = t$, $x(t) = x$ lies entirely in the domain of definition of the function $f(x, \tau)$.

2.2. Let $X^{(0)}[\tau, t]$ be the fundamental matrix of system (2.5) for $w \equiv 0$. Let us consider the quantity

$$\zeta(\tau) = \|l'\{X^{(0)}[\vartheta, \tau]\,B(\tau)\}_m\| \tag{2.6}$$

For any unit vector l $(\|l\| = 1)$ the function $\zeta(\tau)$ (2.6) can vanish only at a finite number of points τ_j $(j = 1, \ldots, k)$ in the interval $[t, \vartheta]$; moreover,

$$|[d\xi(\tau)/d\tau]_{\tau=\tau_j}| \geqslant k_1 > 0 \qquad (k_1 = \text{const})$$

Theorem 2.1. Let Conditions 2.1, 2.2 be fulfilled. Then for $\lambda \leqslant \lambda_0$ (λ_0 is sufficiently small) the motion $x^\circ(\tau; w^\circ) = x^\circ(\tau; l, t, x, \lambda)$ of system (2.1) which satisfies condition (2.3) is generated by the control $w^\circ(\tau) = w^\circ(\tau; l, t, x, \lambda)$ defined by the maximum condition

$$\int_t^\vartheta l'\{X[\vartheta, \tau; l, t, x, \lambda]\,B(\tau)\}_m w^\circ(\tau)\,d\tau =$$

$$= \max_{\|w\| \leqslant \zeta} \int_t^\vartheta l'\{X[\vartheta, \tau; l, t, x, \lambda]\,B(\tau)\}_m\, w(\tau)\,d\tau \tag{2.7}$$

We can prove this theorem by means of the following iterative process. We begin by computing the motion $x^{(0)}(\tau; w)$ of system (2.1) for $\lambda = 0$ and $w = w(\tau)$ satisfying condition (2.2). For $\tau = \vartheta$ we obtain

$$x^{(0)}(\vartheta; w) = X^{(0)}[\vartheta, t]\,x + \int_t^\vartheta X^{(0)}[\vartheta, \tau]\,B(\tau)\,w(\tau)\,d\tau$$

Let us set

$$\rho^{(0)}[l, t, x] = \max_{\|w\|\leqslant\zeta} l'\{x^{(0)}(\vartheta; w)\}_m =$$

$$= l'\{X^{(0)}[\vartheta, t]\, x\}_m + \max_{\|w\|\leqslant\zeta} \int_t^{\vartheta} l'\{X^{(0)}[\vartheta, \tau]\, B(\tau)\}_m\, w(\tau)\, d\tau \tag{2.8}$$

Let $w^{(0)}(\tau; l, t, x)$ be the control which maximizes the second term in (2.8) and let $x^{(0)}(\tau; l, t, x, \lambda)$ be the motion of the system (2.1) for $w = w^{(0)}(\tau; l, t, x)$ under the initial condition $\tau = t$, $x^{(0)}(t; l, t, x, \lambda) = x$. Let us construct the equations in variations

$$\frac{d\delta x^{(1)}}{d\tau} = A^{(1)}(\tau: l, t, x, \lambda)\, \delta x^{(1)} + B(\tau)\, \delta w \tag{2.9}$$

for system (2.1) along the motion $x^{(0)}(\tau; l, t, x, \lambda)$ and denote the fundamental matrix of system (2.9) for $\delta w \equiv 0$ by $X^{(1)}[\vartheta, \tau; l, t, x, \lambda]$. Then

$$x^{(1)}(\tau; l, t, x, \lambda) = x^{(0)}(\tau; l, t, x, \lambda) - \tag{2.10}$$

$$- \int_t^{\tau} X^{(1)}[\tau, \xi; l, t, x, \lambda]\, B(\xi)\, w^{(0)}(\xi; l, t, x)\, d\xi +$$

$$+ \int_t^{\tau} X^{(1)}[\tau, \xi; l, t, x, \lambda]\, B(\xi)\, (w^{(0)}(\xi; l, t, x) + \delta w(\xi))\, d\xi$$

$$\rho^{(1)}[l, t, x, \lambda] = l'\{x^{(0)}(\vartheta; l, t, x, \lambda)\}_m -$$

$$- \int_t^{\vartheta} l'\{X^{(1)}[\vartheta, \tau; l, t, x, \lambda]\, B(\tau)\}_m\, w^{\circ}(\tau; l, t, x)\, d\tau +$$

$$+ \max_{\|w\|\leqslant\zeta} \int_t^{\vartheta} l'\{X^{(1)}[\vartheta, \tau; l, t, x, \lambda]\, B(\tau)\}_m\, w(\tau)\, d\tau$$

Let the last term in (2.10) be maximized by the control $w^{(1)}(\tau; l, t, x, \lambda)$. Continuing this process, we obtain the sequence of controls $w^{(k)}(\tau; l, t, x, \lambda)$ and the corresponding sequence of motions $x^{(k)}(\tau; l, t, x, \lambda)$ of system (2.1). When Conditions 2.1, 2.2 are fulfilled and $\lambda \leqslant \lambda_0$, the sequence $w^{(k)}$ converges in measure to the control $w^{\circ}(\tau; l, t, x, \lambda)$ ($\|w^{\circ}\| \leqslant \zeta$) which satisfies maximum condition (2.7); the corresponding sequence of motions $x^{(k)}$ converges uniformly to the motion $x^{\circ}(\tau; l, t, x, \lambda)$ of system (2.1) generated by the control $w^{\circ}(\tau; l, t, x, \lambda)$ [6, 7].

Now let us consider in the m-dimensional space $\{q\}$ of points $q = \{x\}_m$ the attainability domain $G(\vartheta, t, x, \lambda)$ of system (2.1) from the state $x(t) = x$ up to the instant $\tau = \vartheta$. From now on we shall consider only those cases where the attainability domain $G(\vartheta, t, x, \lambda)$ is convex. In such cases the quantity $\rho[l, t, x, \lambda]$ (2.3) by definition describes the support function of the convex set $G(\vartheta, t, x, \lambda)$. This implies the validity of the following statement.

Theorem 2.2. Let Conditions 2.1, 2.2 be fulfilled and let the attainability domain $G(\vartheta, t, x, \lambda)$ of system (2.1) from the state $x(t) = x$ up to the instant $\tau = \vartheta$ be convex for $\lambda \leqslant \lambda_0$. The domain $G(\vartheta, t, x, \lambda)$ for $\lambda \leqslant \lambda_0$ can be described as follows: it is the set of points q in the m-dimensional space $\{q\}$ for which the inequality

$$\rho[l, t, x, \lambda] - l'q \geqslant 0 \tag{2.11}$$

holds for all unit vectors l ($\|l\| = 1$). The quantity $\rho[l, t, x, \lambda]$ appearing in the left side of inequality (2.11) can be determined from condition (2.3).

3. The regular case. Let the position $y[t] = y$, $z[t] = z$ be realized by the instant t. Let us consider in the m-dimensional space $\{q\}$ of points $q = \{y\}_m$ and $q = \{z\}_m$ the attainability domains $G^{(1)}(\vartheta, t, y, \lambda)$ and $G^{(2)}(\vartheta, t, z, \lambda)$ for the motions $y(\tau)$ (1.1) and $z(\tau)$ (1.2) from the states $y(t) = y$, $z(t) = z$ up to the instant $\tau = \vartheta$ under restrictions (1.3). The symbol $G_\varepsilon^{(1)}(\vartheta, t, y, \lambda)$ denotes the closed Euclidean ε-neighborhood of the domain $G^{(1)}(\vartheta, t, y, \lambda)$. Let $\varepsilon^\circ(t, y, z, \lambda)$ be the smallest $\varepsilon \geqslant 0$ for which

$$G^{(2)}(\vartheta, t, z, \lambda) \subset G_\varepsilon^{(1)}(\vartheta, t, y, \lambda) \tag{3.1}$$

Let us suppose that the following conditions are fulfilled.

Condition 3.1. The motions of systems (1.1) and (1.2) for $\lambda = 0$ generated by all the possible controls subject to restrictions (1.3) under the initial conditions $y_0 \in \Gamma_1^\circ$, $z_0 \in \Gamma_2^\circ$ lie entirely in the domains Γ_1 and Γ_2.

Let $Y^{(0)}[\vartheta, \tau]$ and $Z^{(0)}[\vartheta, \tau]$ be the fundamental matrices of Eqs. (1.1) and (1.2) for $\lambda = 0$, $u \equiv 0$, $v \equiv 0$.

Let us set

$$\xi_1(\tau) = \| l' \{Y^{(0)}[\vartheta, \tau] B^{(1)}(\tau)\}_m \|$$

$$\xi_2(\tau) = \| l' \{Z^{(0)}[\vartheta, \tau] B^{(2)}(\tau)\}_m \| \tag{3.2}$$

Condition 3.2. Whatever the unit vector l ($\| l \| = 1$), the functions $\xi_1(\tau)$, $\xi_2(\tau)$ (3.2) can vanish only at a finite number of points $\tau_j^{(1)}$ and $\tau_j^{(2)}$ from the segment $[t, \vartheta]$; moreover,

$$|[d\xi_1(\tau)/d\tau]_{\tau=\tau_j^{(1)}}| \geqslant k_1 > 0, \qquad |[d\xi_2(\tau)/d\tau]_{\tau=\tau_j^{(2)}}| \geqslant k_2 > 0$$
$$(k_1, k_2 = \mathrm{const})$$

Condition 3.3. The attainability domains $G^{(1)}(\vartheta, t, y, \lambda)$ and $G^{(2)}(\vartheta, t, z, \lambda)$ are convex for $\lambda \leqslant \lambda_0$.

Lemma 3.1. Let Conditions 3.1–3.3 be fulfilled. Then the smallest $\varepsilon \geqslant 0$ (in the case $\lambda \leqslant \lambda_0$) for which inclusion (3.1) is valid can be determined from the relation

$$\varepsilon^\circ(t, y, z, \lambda) = \max_{\|l\|=1} \{\rho^{(2)}[l, t, z, \lambda] - \rho^{(1)}[l, t, y, \lambda]\} \tag{3.3}$$

where $\rho^{(1)}$ and $\rho^{(2)}$ are the support functions constructed for the pursuing and pursued objects, respectively, i.e.

$$\rho^{(1)} = \max_{\|u\| \leqslant \mu} l' \{y(\vartheta; u)\}_m = l' \{y^\circ(\vartheta; l, t, y, \lambda)\}_m \tag{3.4}$$

$$\rho^{(2)} = \max_{\|v\| \leqslant \nu} l' \{z(\vartheta; v)\}_m = l' \{z^\circ(\vartheta; l, t, z, \lambda)\}_m \tag{3.5}$$

In fact, Conditions 3.1–3.3 and Theorem 2.2 imply that the domains $G_\varepsilon^{(1)}(\vartheta, t, y, \lambda)$ and $G^{(2)}(\vartheta, t, z, \lambda)$ can be constructed as the sets of points q in the m-dimensional space $\{q\}$ for which the inequalities

$$\varepsilon + \rho^{(1)}[l, t, y, \lambda] - l'q \geqslant 0, \quad \rho^{(2)}[l, t, z, \lambda] - l'q \geqslant 0 \tag{3.6}$$

are valid for every unit vector l ($\| l \| = 1$).

Inclusion (3.1) is valid if and only if every point q which satisfies the first inequality of (3.6) for all l ($\| l \| = 1$) also satisfies the second inequality of (3.6). This can happen if and only if

$$\varepsilon + \rho^{(1)}[l, t, y, \lambda] - \rho^{(2)}[l, t, z, \lambda] \geqslant 0 \tag{3.7}$$

for every unit vector l.

Let us take an arbitrary point q from $G^{(2)}$. The second inequality of (3.6) is satisfied by this point for all l. But this automatically implies that this point q satisfies the first inequality of (3.6) for all l, i.e. $q \in G_\varepsilon^{(1)}$.

Now let us verify the necessity of condition (3.7). We can do this by assuming the opposite, i.e. that

$$\varepsilon + \rho^{(1)}[l^*, t, y, \lambda] - \rho^{(2)}[l^*, t, z, \lambda] < 0 \tag{3.8}$$

for some $l = l^*$.

Let us choose a point $q = q^*$ lying on the boundary of the domain $G^{(2)}$ for which the equation

$$\rho^{(2)}[l^*, t, z, \lambda] - l^{*\prime} q = 0 \tag{3.9}$$

is valid.

Such a point $q^* \in G^{(2)}$ necessarily exists by virtue of Theorem 2.1. But the point q^* cannot lie in $G_\varepsilon^{(1)}$, since (3.9) and (3.8) imply that

$$\varepsilon + \rho^{(1)}[l^*, t, y, \lambda] - l^{*\prime} q < 0 \tag{3.10}$$

This inequality contradicts (3.6). Hence, inequality (3.8) cannot hold for any l ($\| l \| = 1$).

Thus, inequality (3.7) is the necessary and sufficient condition for the fulfillment of inclusion (3.1). Completing the proof of Lemma 3.1, we finally note that Eq. (3.3) follows directly from (3.7).

In this section we are concerned with the regular case [4, 5], i.e. with the case where the maximum in the right side of (3.3) is attained on the unit vector $l^\circ = l^\circ(t, y, z, \lambda)$ for all those positions $\{t, y, z\}$ for which $\varepsilon^\circ(t, y, z, \lambda) > 0$.

In the regular case the extremal strategies U_e and V_e are defined by the sets $U_e^*(t, y, z, \lambda)$ and $V_e^*(t, y, z, \lambda)$ of the following form [4, 5].

Definition 3.1. If $\varepsilon^\circ(t, y, z, \lambda) > 0$, then the sets $U_e^*(t, y, z, \lambda)$ and $V_e^*(t, y, z, \lambda)$ consist of all those vectors u_e and v_e which satisfy the conditions

$$l^{\circ\prime}\{Y[\vartheta, t; l^\circ, t, y, \lambda] B^{(1)}(t)\}_m u_e = \max_{u \in U^*} l^{\circ\prime}\{Y[\vartheta, t; l^\circ, t, y, \lambda] B^{(1)}(t)\}_m u \tag{3.11}$$

$$l^{\circ\prime}\{Z[\vartheta, t; l^\circ, t, z, \lambda] B^{(2)}(t)\}_m v_e = \max_{v \in V^*} l^{\circ\prime}\{Z[\vartheta, t; l^\circ, t, z, \lambda] B^{(2)}(t)\}_m v \tag{3.12}$$

where Y and Z are the fundamental matrices of the systems of equations in variations

$$d\delta y/d\tau = A^{(1)\circ}(\tau; l, t, y, \lambda)\delta y, \quad d\delta z/d\tau = A^{(2)\circ}(\tau; l, t, z, \lambda)\delta z$$

constructed for Eqs. (1.1) and (1.2), respectively, along the motions $y^\circ(\tau; l, t, y, \lambda)$ and $z^\circ(\tau; l, t, z, \lambda)$ satisfying Eqs. (3.4) and (3.5).

Definition 3.2. If $\varepsilon^\circ(t, y, z, \lambda) = 0$, then

$$U_e^*(t, y, z, \lambda) = U^*, \qquad V_e^*(t, y, z, \lambda) = V^*$$

We note that (3.11) and (3.12) have the following implications:

a) at instants t when $\| l^{\circ\prime}\{YB^{(1)}\}_m \| \neq 0$ and $\| l^{\circ\prime}\{ZB^{(2)}\}_m \| \neq 0$ the sets $U_e^*(t, y, z, \lambda)$ and $V_e^*(t, y, z, \lambda)$ consist of the single points $u_e[t]$ and $v_e[t]$, where

$$u_e[t] = \mu \frac{(l^{\circ\prime}\{Y[\vartheta, t; l^\circ, t, y, \lambda] B^{(1)}(t)\}_m)'}{\| l^{\circ\prime}\{Y[\vartheta, t; l^\circ, t, y, \lambda] B^{(1)}(t)\}_m \|}$$

$$v_e[t] = \nu \frac{(l^{\circ\prime}\{Z[\vartheta, t; l^\circ, t, z, \lambda] B^{(2)}(t)\}_m)'}{\| l^{\circ\prime}\{Z[\vartheta, t; l^\circ, t, z, \lambda] B^{(2)}(t)\}_m \|}$$

b) we assume that

$$U_e^*(t, y, z, \lambda) = U^*, \qquad V_e(t, y, z, \lambda) = V^*$$

at the instants when

$$\| l^{\circ\prime}\{YB^{(1)}\}_m \| = 0, \qquad \| l^{\circ\prime}\{ZB^{(2)}\}_m \| = 0$$

In the regular case the extremal strategies are permissible [4, 5, 9] and the following statements are valid:

Theorem 3.1. Let Conditions 3.1–3.3 be fulfilled and let the regular case hold. The extremal strategy U_e for $\lambda \leqslant \lambda_0$ is then the optimal strategy which solves Problem 1.1. Here

$$(\gamma[\vartheta] \mid t_0, y_0, z_0, U_e, v) \leqslant \varepsilon^\circ(t_0, y_0, z_0, \lambda)$$

for every initial position $y_0 \in \Gamma_1^\circ$ and $z_0 \in \Gamma_2^\circ$ and for every permissible realization $v[t]$ of the control v.

Theorem 3.2. Let Conditions 3.1–3.3 be fulfilled and let the regular case hold. The extremal strategy V_e for $\lambda \leqslant \lambda_0$ is then the optimal strategy which solves Problem 1.2. Here

$$(\gamma[\vartheta] \mid t_0, y_0, z_0, u, V_e) \geqslant \varepsilon^\circ(t_0, y_0, z_0, \lambda)$$

whatever the initial position $y_0 \in \Gamma_1^\circ$ and $z_0 \in \Gamma_2^\circ$ and whatever the permissible realization $u[t]$ of the control u.

To prove Theorems 3.1 and 3.2 we must investigate the behavior of the derivative $d\varepsilon^\circ[t]/dt$ of the function $\varepsilon^\circ[t] = \varepsilon^\circ(t, y[t], z[t], \lambda)$ for $\varepsilon^\circ[t] > 0$, $t < \vartheta$ along the motions $y[t]$ (1.1) and $z[t]$ (1.2) generated by the strategies U_e, V (or U, V_e).

To compute the derivative $d\varepsilon^\circ[t]/dt$ of the absolutely continuous function $\varepsilon^\circ[t]$ we make use of the following considerations [8]. In the regular case the vector $l^\circ(t, y, z, \lambda)$ which maximizes the right side of Eq. (3.3) depends continuously on t, y, z, λ in the domain $\varepsilon^\circ(t, y, z, \lambda) > 0$. $t_0 \leqslant t < \vartheta$, $\lambda \leqslant \lambda_0$. The continuity of the vector l° [8] implies that the derivatives

$$\frac{\partial \varepsilon^\circ}{\partial t} = \frac{\partial \rho^{(2)}[l^\circ, t, z, \lambda]}{\partial t} - \frac{\partial \rho^{(1)}[l^\circ, t, y, \lambda]}{\partial t} \tag{3.13}$$

$$\frac{\partial \varepsilon^\circ}{\partial y_i} = -\frac{\partial \rho^{(1)}[l^\circ, t, y, \lambda]}{\partial y_i}, \qquad \frac{\partial \varepsilon^\circ}{\partial z_i} = \frac{\partial \rho^{(2)}[l^\circ, t, z, \lambda]}{\partial z_i}$$

exist in the domain $\varepsilon^\circ > 0$, $t_0 \leqslant t < \vartheta$, $\lambda \leqslant \lambda_0$.

Moreover, since the vector l° maximizes the right side of (3.3), we ignore the dependence of the vector l° on t, y, z in computing derivatives (3.13). Making use of the rules of differentiation of the solutions $y^\circ(\tau; l^\circ, t, y, \lambda)$ (3.4) and $z^\circ(\tau; l^\circ, t, z, \lambda)$ (3.5) with respect to the initial data and with respect to the parameter [10], we can show that

$$\frac{d\varepsilon^\circ[t]}{dt} = \frac{\partial \varepsilon^\circ}{\partial t} + \sum_{i=1}^{n} \frac{\partial \varepsilon^\circ}{\partial y_i} \frac{dy_i[t]}{dt} + \sum_{i=1}^{n} \frac{\partial \varepsilon^\circ}{\partial z_i} \frac{\partial z_i[t]}{dt} =$$

$$= \max_{u\in U^*} l^{\circ\prime}\{Y[\vartheta, t; l^\circ, t, y, \lambda] B^{(1)}(t)\}_m u -$$

$$- l^{\circ\prime}\{Y[\vartheta, t; l^\circ, t, y, \lambda] B^{(1)}(t)\}_m u[t] - \max_{v\in V} l^{\circ\prime}\{Z[\vartheta, t; l^\circ, t, z, \lambda] B^{(2)}(t)\}_m v +$$

$$+ l^{\circ\prime}\{Z[\vartheta, t; l^\circ, t, z, \lambda] B^{(2)}(t)\}_m v[t] \tag{3.14}$$

for almost all t.

Expression (3.14) implies that $d\varepsilon^\circ[t]/dt \leqslant 0$ for $\varepsilon^\circ[t] > 0$ for almost all t if the pursuer maintains the extremal strategy U_e while the target maintains an arbitrary permissible strategy V. Conversely, if the target maintains the extremal strategy V_e while the pursuer deviates from the extremal strategy U_e, then $d\varepsilon^\circ[t]/dt \geqslant 0$ for $\varepsilon^\circ[t] > 0$ for almost all t. This implies the validity of Teorems 3.1 and 3.2.

Theorems 3.1 and 3.2 imply that the games problem of convergence of quasilinear objects has a saddle point in the regular case.

Theorem 3.3. Let Conditions 3.1–3.3 be fulfilled and let the regular case hold. The extremal strategies U_e and V_e then yield the saddle point of the convergence game, i.e.

$$(\gamma[\vartheta] \mid t_0, y_0, z_0, U_e, v) \leqslant (\gamma[\vartheta] \mid t_0, y_0, z_0, U_e, V_e) \leqslant (\gamma[\vartheta] \mid t_0, y_0, z_0, u, V_e)$$

for every initial position $y_0 \in \Gamma_1^\circ$ and $z_0 \in \Gamma_2^\circ$.

4. Example. Let the behavior of the pursuer and target be described by the equations

$$\begin{array}{lllll} \dot y_1 = y_2, & \dot y_2 = \lambda y_2^2 + u_1, & \dot y_3 = y_4, & \dot y_4 = u_2, & u_1^2[t] + u_2^2[t] \leqslant \mu^2 \\ \dot z_1 = z_2, & \dot z_2 = \lambda z_2^2 + v_1, & \dot z_3 = z_4, & \dot z_4 = v_2, & v_1^2[t] + v_2^2[t] \leqslant \nu^2 \end{array} \tag{4.1}$$

and let

$$\gamma[\vartheta] = [(y_1(\vartheta) - z_1(\vartheta))^2 + (y_3(\vartheta) - z_3(\vartheta))^2]^{1/2}$$

It is easy to verify that Conditions 3.1 – 3.3 hold for Eqs. (4.1). Carrying out the computations in accordance with the procedure of Sect. 2, we find that the support functions $\rho^{(1)}$ and $\rho^{(2)}$ are described by the equations

$$\rho^{(1)} = {}^1/_2\,\mu(\vartheta - t)^2 + l_1(y_1 + (\vartheta - t)y_2) + l_3(y_3 + (\vartheta - t)y_4) + {}^1/_6\,\lambda l_1(\vartheta -$$
$$- t)^2\{3y_2^2 + 2\mu y_2(l_1 - l_3^2)(\vartheta - t) - \mu^2(\vartheta - t)^2 l_1({}^5/_2 l_1 + l_3^2)\} + \ldots$$

$$\rho^{(2)} = {}^1/_2\nu(\vartheta - t)^2 + l_1(z_1 + (\vartheta - t)z_2) + l_3(z_3 + (\vartheta - t)z_4) + {}^1/_6\lambda l_1(\vartheta -$$
$$- t)^2\{3z_2^2 + 2\nu z_2(l_1 - l_3^2)(\vartheta - t) - \nu^2(\vartheta - t)^2 l_1({}^5/_2 l_1 + l_3^2)\} + \ldots$$

Let us introduce the notation $x_i = y_i - z_i$, $\zeta = \mu - \nu$. Then

$$\varepsilon^\circ(t, y, z, \lambda) = \max_D\{- l_1(x_1 + (\vartheta - t)x_2) - l_3(x_3 + (\vartheta - t)x_4) - {}^1/_2\zeta(\vartheta - t)^2 +$$
$$+ \lambda\cdot {}^1/_6\, l_1(\vartheta - t)^2[3z_2^2 - 3y_2^2 + 2(\nu z_2 - \mu y_2)(l_1 - l_3^2)(\vartheta - t) - l_1(\vartheta - \tag{4.2}$$
$$- t)^2(\mu^2 - \nu^2)({}^5/_2 l_1 + l_3^2)] + \ldots\} \quad (D \equiv l_1^2 + l_3^2 = 1)$$

For $\lambda \leqslant \lambda_0$ the attainability domains of objects (4.1) are convex and nearly disk-shaped. Hence, for those positions where $\varepsilon^\circ > 0$ the maximum in the right side of (4.2) is attained on the unique vector l°. This means that the regular case holds. Here

$$l_1^\circ = l_1^{(0)} + \lambda l_1^{(1)} + \ldots, \qquad l_3^\circ = l_3^{(0)} + \lambda l_3^{(1)} + \ldots$$

$$l_1^{(0)} = -\frac{x_1 + (\vartheta - t)x_2}{[(x_1 + (\vartheta - t)x_2)^2 + (x_3 + (\vartheta - t)x_4)^2]^{1/2}}$$

$$l_3^{(0)} = -\frac{x_3 + (\vartheta - t)\,x_4}{[(x_1 + (\vartheta - t)\,x_2)^2 + (x_3 + (\vartheta - t)\,x_4)^2]^{1/2}}$$

$$l_1^{(1)} = \frac{l_3^{(0)}\,(l_3^{(0)}N_1 - l_1^{(0)}N_2)}{[(x_1 + (\vartheta - t)\,x_2)^2 + (x_3 + (\vartheta - t)\,x_4)^2]^{1/2}}$$

$$l_3^{(1)} = \frac{l_1^{(0)}\,(l_1^{(0)}N_2 - l_3^{(0)}N_1)}{[(x_1 + (\vartheta - t)\,x_2)^2 + (x_3 + (\vartheta - t)\,x_4)^2]^{1/2}}$$

$$N_1 = {}^1/_6\,(\vartheta - t)^2\,[3\,(z_2^2 - y_2^2) + 2\,(\nu z_2 - \mu y_2)\,(\vartheta - t)\,(2l_1^{(0)} - (l_3^{(0)})^2) + {}^{15}/_2\,(\mu^2 - \nu^2)\,(l_1^{(0)})^2\,(\vartheta - t)^2 + 2\,(\mu^2 - \nu^2)\,l_1^{(0)}\,(l_3^{(0)})^2\,(\vartheta - t)^2]$$

$$N_2 = {}^1/_3\,(\vartheta - t)^3\,[-2l_3^{(0)}\,(\nu z_2 - \mu y_2) + (\mu^2 - \nu^2)\,l_1^{(0)}l_3^{(0)}\,(\vartheta - t)]$$

The extremal strategies U_e and V_e can be described as follows:

1) if $\varepsilon^\circ\,(t, y, z, \lambda) > 0$, then the sets $U_e^*\,(t, y, z, \lambda)$ and $V_e^*\,(t, y, z, \lambda)$ consist of the single points $u_e\,[t]$ and $v_e\,[t]$, where

$$u_{e1}\,[t] = \mu\,[l_1^{(0)} + \lambda l_1^{(1)} + \ldots], \quad v_{e1}\,[t] = \nu\,[l_1^{(0)} + \lambda l_1^{(1)} + \ldots]$$

$$u_{e2}\,[t] = \mu\,\{l_3^{(0)} + \lambda\,[l_3^{(1)} - l_1^{(0)}l_3^{(0)}\,((\vartheta - t)\,y_2 + {}^1/_3\mu l_1^{(0)}\,(\vartheta - t)^2)] + \ldots\}$$

$$v_{e2}\,[t] = \nu\,\{l_3^{(0)} + \lambda\,[l_3^{(1)} - l_1^{(0)}l_3^{(0)}\,((\vartheta - t)\,z_2 + {}^1/_3\nu l_1^{(0)}\,(\vartheta - t)^2)] + \ldots\}$$

2) if $\varepsilon^\circ\,(t, y, z, \lambda) = 0$, then

$$U_e^*\,(t, y, z, \lambda) = U^*, \quad V_e^*\,(t, y, z, \lambda) = V^*$$

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BIBLIOGRAPHY

1. Isaacs, R., Differential Games. Moscow, "Mir", 1967.
2. Petrosian, L. A., A dynamic pursuit game in the presence of friction forces. Dokl. Akad. Nauk ArmSSR Vol. 44, №1, 1967.
3. Gnoenskii, L. S., On the tracking problem. PMM Vol. 26, №5, 1962.
4. Krasovskii, N. N., A differential convergence game. Differentsial'nye Uravneniia Vol. 5, №3, 1969.
5. Krasovskii, N. N., Games Problems on the Encounter of Motions. Moscow, "Nauka", 1970.
6. Krasovskii, N. N., Theory of Motion Control. Moscow, "Nauka", 1968.
7. Al'brekht, E. G., On optimal control of the motion of quasilinear systems. Differentsial'nye Uravneniia Vol. 5, №3, 1969.
8. Krasovskii, N. N. and Subbotin, A. I., Optimal strategies in a linear differential game. PMM Vol. 33, №4, 1969.
9. Filippov, A. F., Differential equations with a discontinuous right side. Mat. Sb. Vol. 51(93), №1, 1960.
10. Stepanov, V. V., A Course of Differential Equations. Moscow, Fizmatgiz, 1959.

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